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New Algorithms for One-Loop Integrals

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Abstract

New algorithms are presented for evaluating the scalar one loop integrals for three- and four-point functions for arbitrary masses and external momenta. These formulations are useful both for analytic integration and for numerical evaluation in a computer program. The expressions are very compact and provide for an easy isolation of asymptotic behaviour and potential numerical problems.

The tensor integrals have also been rewritten according to new algorithms, making it very easy to express amplitudes in terms of scalar loop integrals.

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New algorithms are presented for evaluating the scalar one loop integrals for three- and four-point functions for arbitrary masses and external momenta. These formulations are useful both for analytic integration and for numerical evaluation in a computer program. The expressions are very compact and provide for an easy isolation of asymptotic behaviour and potential numerical problems.

The tensor integrals have also been rewritten according to new algorithms, making it very easy to express amplitudes in terms of scalar loop integrals.

1 Introduction

The evaluation of scalar loop integrals is one of the time consuming parts of radiative correction computations in high energy physics. Of course there exists a general solution for these integrals [1], but the formulae are involved and also the implementation in a computer program is far from straightforward. This is particularly so due to very complicated numerical problems. The traditional way to avoid these problems is to do the integrals by hand for each special case separately (special cases can give much compacter formulae) and use a rather large precision when programming these formulae. This costs much time. M. Veltman on the other hand has programmed the general formulae in a program called FormF which allowed him to do some very complicated radiative correction computations [2]. On a restricted class of computers this program is available. But even in this program the numerical problems have not been solved completely, even though the precision with which some intermediate results are evaluated is sometimes 120 decimal digits! There are many calculations for which it would be difficult to use FormF. In addition the use of such an extended precision makes the evaluation of the integrals rather slow. As long as one considers corrections to reactions with only two particles in the final state this is not much of a problem. The amount of computer time that is needed for the computation of reactions with more particles in the final state seems however prohibitive.

It is therefore necessary to study the original integrals for scalar three- and four-point functions again in order to find expressions for them that are numerically stable to such an extent that they can be programmed in a standard precision. This involves of course more work. During this work we encountered some rather striking ways to rewrite the integrals, allowing us to classify the quantities that cause the numerical problems either as kinematical determinants or as the analytic continuations of kinematic determinants that are part of the computation of the corresponding radiative diagrams. The occurrence of these determinants should not be a great surprise. Current conservation and gauge cancellations should occur as a function of the kinematic quantities that are present in a problem. This means that there must be objects that harbour great cancellations (for a high energy t-channel reaction 10 digits cancellations are not uncommon). Those objects can usually be written either as the sum of terms with vector products (dotproducts) and masses — the unstable form — or the combination of a number of Levi-Civita tensors — the stable form — [3]. This should not be very surprising: the electromagnetic field-strength tensor $F^{\mu\nu}$ is antisymmetric in μ and ν and can also be written as a combination of two Levi-Civita tensors. Furthermore, the easiest way to define the dual field-strength $\tilde{F}^{\mu\nu}$ is through a single Levi-Civita tensor.

The combination of a pair of Levi-Civita tensors is often a way to write a determinant as in the case of Gram determinants [4,5]. Very often the value of these determinants is already present when the kinematic configuration is constructed. Hence we will try to express many quantities in terms of these determinants, thereby solving most numerical problems. In some cases we will give some examples of the cancellations that may occur if the proper precautions are omitted. In general it is best to use the same rewrites of

the integrals that we use to obtain a stable configuration also for analytic calculations. Only when the answer is in a stable form is it possible to see how the answer depends on its parameters. This is particularly important for the study of asymptotic behaviour.

In actual calculations one will encounter usually also integrals which have powers of the loop momentum (the momentum that is integrated over) in the numerator. Such integrals are called tensor integrals. It has been shown that such integrals can be reduced to a linear combination of scalar loop integrals [6,2]. The reduction scheme that is given in the literature is most suited for numerical application, but even so it is rather ill adapted with respect to t-channel reactions. The kinematic determinants are used in their lengthy unstable form [7]. Again the use of Levi-Civita tensors can improve their properties considerably. We present new algorithms for the reduction scheme that can be applied much easier in an analytic calculation. They make the reduction to an analytic expression in terms of scalar loop integrals rather easy, especially with the use of a computer algebra program like Schoonschip [8] or Form [9].

The layout of the paper is as follows. First we will discuss our notations and the definitions of the various determinants we will be using after which we give a brief discussion of numerical instabilities. This allows us then to give a rather straight forward solution for the three-point function integral. Next we solve the four point integral, using the reduction scheme that expresses it in terms of two three-point functions. This algorithm [1] can be used for nearly all reactions. Finally we address the problem of the reduction of tensor integrals.

2 Definitions and Notations

2.1 Kinematical determinants

An algebraically compact and numerically stable notation for tree-level matrix elements can often be obtained by the use of kinematical determinants [3,4,5], which are also called Gram determinants. These are expressions of the form

$$\det(A) = \det \begin{pmatrix} p_1 \cdot q_1 & \cdots & p_1 \cdot q_n \\ \vdots & \ddots & \vdots \\ p_n \cdot q_1 & \cdots & p_n \cdot q_n \end{pmatrix} \quad (1)$$

One way to write determinants is with the use of Levi-Civita tensors. Most of our formulae will not be sensitive to the choice of metric and hence be also insensitive to the exact definition of the Levi-Civita tensors. Whenever we use Levi-Civita tensors they will be occurring in pairs and all indices will be contracted, either with indices of other Levi-Civita tensors or with indices of four vectors. For a numerical evaluation one may use the convention $\varepsilon^{0123} = -\varepsilon_{0123} = +1$ (in Bjorken and Drell metric). The determinant of the $n \times n$ matrix A can be written as

$$\det(A) = p_{1\alpha_1} p_{2\alpha_2} \cdots p_{n\alpha_n} \varepsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} \varepsilon_{\beta_1 \beta_2 \cdots \beta_n} q_1^{\beta_1} q_2^{\beta_2} \cdots q_n^{\beta_n} \quad (2)$$

The contraction of the Levi-Civita tensors to give $n!$ terms with n vector products (dotproducts) each and the substitution $p_i \cdot q_j = A_{ij}$ gives then the more familiar form of the determinant written out in terms of the components of A . The above form of the determinant can be written more compactly using Schoonschip notation:

$$\det(A) = \varepsilon^{p_1 p_2 \dots p_n} \varepsilon_{q_1 q_2 \dots q_n} \quad (3)$$

We have replaced indices that are contracted with the index of a vector by that vector, thereby gaining much clarity in the notation. When such an index has been contracted it doesn't matter any more whether the vector is placed as an upper index or as a lower index. This allows us to use an even more compact notation with the use of generalised Kronecker deltas:

$$\det(A) = \delta_{q_1 \dots q_n}^{p_1 \dots p_n} \quad (4)$$

When a generalised Kronecker delta has fewer momenta than the dimension of the vector space in which the vectors are defined the corresponding Levi-Civita tensors are saturated with indices that are common to the two Levi-Civita tensors. The normalisation is always one:

$$\delta_{q_1 \dots q_m}^{p_1 \dots p_m} = \varepsilon^{p_1 \dots p_m \alpha_{m+1} \dots \alpha_n} \varepsilon_{q_1 \dots q_m \alpha_{m+1} \dots \alpha_n} / \Gamma(n - m + 1) \quad (5)$$

In terms of dotproducts this gives the same result as when the extra indices are ignored and the temporary assumption is made that the dimension of the vectors is identical to the number of vectors:

$$\delta_{q_1 q_2}^{p_1 p_2} = \varepsilon^{p_1 p_2 \alpha_3 \dots \alpha_n} \varepsilon_{q_1 q_2 \alpha_3 \dots \alpha_n} / \Gamma(n - 1) \quad (6)$$

$$= \varepsilon^{p_1 p_2} \varepsilon_{q_1 q_2} \quad (7)$$

$$= p_1 \cdot q_1 p_2 \cdot q_2 - p_1 \cdot q_2 p_2 \cdot q_1 \quad (8)$$

An example of such a kinematical determinant is the 2×2 determinant associated with a three-point function: $\delta_{p_1 p_2}^{p_1 p_2} = p_1^2 p_2^2 - (p_1 \cdot p_2)^2$. In a two body decay the center of mass momentum of the decay particles is given by $p = \sqrt{-\delta_{p_1 p_2}^{p_1 p_2}} / M$ (with M the mass of the decaying particle.)

With the use of the above conventions it is also possible to deal with indices in n dimensions for a non-integer value of n . Let us assume that μ and ν are indices in n dimensions, then

$$\delta_{q_1 \dots q_m \nu}^{p_1 \dots p_m \mu} = \varepsilon^{p_1 \dots p_m \mu \alpha_{m+2} \dots \alpha_n} \varepsilon_{q_1 \dots q_m \nu \alpha_{m+2} \dots \alpha_n} / \Gamma(n - m) \quad (9)$$

The occurrence of α_n should be seen as a 'formal' notation. When this equation is multiplied by g_μ^ν we obtain

$$\begin{aligned} \delta_{q_1 \dots q_m \mu}^{p_1 \dots p_m \mu} &= \varepsilon^{p_1 \dots p_m \mu \alpha_{m+2} \dots \alpha_n} \varepsilon_{q_1 \dots q_m \mu \alpha_{m+2} \dots \alpha_n} / \Gamma(n - m) \\ &= \varepsilon^{p_1 \dots p_m} \varepsilon_{q_1 \dots q_m} \Gamma(n - m + 1) / \Gamma(n - m) \\ &= (n - m) \varepsilon^{p_1 \dots p_m} \varepsilon_{q_1 \dots q_m} \\ &= (n - m) \delta_{q_1 \dots q_m}^{p_1 \dots p_m} \end{aligned} \quad (10)$$

We will need this property when dealing with the tensor integrals. Another way to derive the above identity is cleaner from the mathematical viewpoint, but less general. We write in analogy to (6) $\delta_{q_1 \dots q_m}^{p_1 \dots p_m \mu \nu}$ in terms of dotproducts and vectors with the indices μ and ν . There will be terms with g_ν^μ and when these are contracted with g_μ^ν they give the dimension, which is n . When everything is added we obtain the result of equation (10).

2.2 Internal vectors

In order to use the above determinants for the evaluation of scalar loop integrals there should be vectors corresponding to the internal lines. There is of course a large freedom choosing such vectors because there is a loop momentum present that is integrated over, which we can shift. We like to have a set of vectors s_i^μ that obey momentum conservation and a quasi on-shell condition:

$$p_i^\mu = s_{i+1}^\mu - s_i^\mu \quad (11)$$

$$s_i^2 = m_i^2 \quad (12)$$

The p_i are the external momentum vectors and the m_i are the masses of the internal lines, numbered such that p_i and p_{i+1} are connected by an internal line with mass m_{i+1} . We have used a notation in which the index i in p_i^μ and in s_i^μ is taken modulus the number of external vectors. This means that for a four-point function $s_5^\mu = s_1^\mu$. Of course the internal vectors s_i are not really on-shell, but when taking into account that these lines can belong to real particles in the corresponding radiative diagrams it should be clear that we are looking at an ‘analytic continuation’ of the domain of validity of the vectors in those diagrams.

If one tries to solve for the components of the s_i one will find that in general some components will have complex values. (One can find a frame in which one is purely imaginary and the others are real.) This turns out to be no problem at all. Our formulae can always be written such that the s_i occur only in four vector products with each other and the p_i . All those products are real:

$$s_{i+1} \cdot s_i = \frac{s_{i+1}^2 + s_i^2 - p_i^2}{2} \quad (13)$$

$$s_i \cdot p_i = \frac{s_{i+1}^2 - s_i^2 - p_i^2}{2} \quad (14)$$

All other dotproducts can be reduced in a similar fashion to sums of real parameters so the imaginary parts of these vectors never occur in a physical quantity.

When we use these vectors they are not only useful in the determinants but they also allow a much easier way to deal with the transformation from the integral over momentum space to the integral over the Feynman parameters [10]:

$$X_0^k = \int_{M_n} \frac{d^n Q}{(Q^2 - m_1^2)((Q + p_1)^2 - m_2^2) \cdots ((Q + p_1 + p_2 + \cdots + p_{k-1})^2 - m_k^2)}$$

$$= \int_{M_n - s_1} \frac{d^n Q}{(Q^2 + 2s_1 \cdot Q)(Q^2 + 2s_2 \cdot Q) \cdots (Q^2 + 2s_k \cdot Q)} \quad (15)$$

$$= i\pi^{\frac{n}{2}} \int_{\substack{x_1 + \cdots + x_{k-1} \leq 1 \\ x_1, \dots, x_{k-1} \geq 0}} dx_1 \cdots dx_{k-1} \Gamma(k - \frac{n}{2}) \\ \times \left\{ [x_1(s_1 - s_k) + \cdots + x_{k-1}(s_{k-1} - s_k) + s_k]^2 - i\epsilon \right\}^{\frac{n}{2} - k} \quad (16)$$

Note that the shift $l^\mu = Q^\mu + (s_1^\mu - s_k^\mu)x_1 + \cdots + s_k^\mu$ after the introduction of the Feynman parameters brings the integration back to the real Minkowski space M_n . So at the onset the vectors s_i give us a handy way to obtain the integrals in the Feynman parameters. In the sequel we will encounter them frequently inside kinematic determinants because although the $s_i - s_k$ in the final integral can be expressed in terms of the external momenta p_i one single vector s_k cannot be removed. Next we could write everything in terms of dotproducts and substitute the expressions (11) and (12), but that would destroy the simplicity of our formulae.

2.3 Evaluation

There are two advantages to using determinants. The first is the compact notation, often allowing formulae that would be very lengthy in their expanded form to be written on a few lines. The other advantage lies in their numerical properties. When these determinants are written in terms of dotproducts they are notorious for their ill numerical behaviour. On the other hand most of the determinants that involve external particles can often only be known to good precision from the kinematics of the problem (e.g. near thresholds). This doesn't hold for the determinants that involve any of the internal vectors s_i . In such a case we may have to use the liberty inherent in a determinant to take any independent set of vectors. For instance if $p_1 + p_2 + p_3 = 0$ we can write $\delta_{\substack{p_1 p_2 \\ p_1 p_2}} = \delta_{\substack{p_2 p_3 \\ p_2 p_3}} = \delta_{\substack{p_3 p_1 \\ p_3 p_1}} = \delta_{\substack{p_1 p_2 \\ p_2 p_3}} = \cdots$. A knowledge of all dotproducts ($p_i \cdot p_j$, $p_i \cdot s_j$ and $s_i \cdot s_j$) is almost always sufficient to obtain an accurate value. We assume that the dotproducts involving only physical momenta are known from the kinematics of the problem. The dotproducts involving internal vectors can easily be calculated from the masses, momenta squared and their differences. (These differences allow us to evaluate for instance $p^2 + s_1^2 - s_2^2$ accurately even if $p^2 \ll s_1^2 \approx s_2^2$).

2.4 Numerical instabilities

There are two basic ways in which a numerical calculation can give an inaccurate answer. The first occurs when the final result depends very strongly on the input parameters ($\partial f / \partial x \gg f/x$). An example is $\log(x)$ for $x \approx 1$. A very small change in x will give a very large relative change in the result. This problem can be solved by switching to more appropriate parameters in this region, for instance $y = 1 - x$. It is therefore often necessary to use an overcomplete set of variables. In the case of the scalar n -point functions it is necessary to include all differences of masses and momenta squared as parameters. Near a threshold even more information may be necessary.

The second possibility concerns a poor choice of the algorithm. A good example is the evaluation of $E - p$ for a 50 GeV electron. Each of the elements in the formula has a magnitude of (about) 50 GeV, but the result is very small. It is therefore better to multiply the numerator and the denominator with $E + p$ and work out the numerator to obtain the (stable) formula $m^2/(E + p)$. Here there are no more cancellations and we can see quickly that the answer is (roughly) $0.25 \cdot 10^{-8}$ GeV which means that in the first formula we would have lost more than 10 digits accuracy! The improvement of formulae by the above method is a rather common procedure in relativistic kinematics. Another famous example is the calculation of the roots of the quadratic equation $ax^2 - 2bx + c = 0$. The standard formula $x^\pm = (b \pm \sqrt{b^2 - ac})/a$ gives one root stably, the other one follows from $x^+x^- = c/a$. A final example is the following series of numerical substitutions

$$\begin{aligned} y &= 1 - x \\ z &= 1 - y \end{aligned}$$

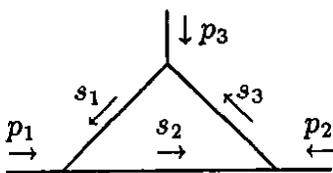
for $x \ll 1$. One should evidently keep the value $x = 1 - y$ apart to obtain a reasonable answer, i.e. retain an overcomplete set of variables at intermediate steps.

The above methods are used throughout the rest of the paper. Specifically, we use determinants and roots of quadratic equations to obtain an overcomplete set of variables with which we can calculate the arguments to the dilogarithms. The cancellations among the dilogarithms are handled by rewriting the equations to a more suitable form, using many of the properties of the dilogarithm [11].

3 The three-point function

3.1 Introduction

The scalar three-point function is given by the scalar diagram



Its corresponding integral is given by

$$\begin{aligned} & C_0(m_1^2, m_2^2, m_3^2, p_1^2, p_2^2, p_3^2) \\ &= \int_{M_4} \frac{d^4Q}{(Q^2 - m_1^2)((Q + p_1)^2 - m_2^2)((Q + p_1 + p_2)^2 - m_3^2)} \end{aligned} \quad (17)$$

$$= i\pi^2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[x(s_1 - s_3) + y(s_2 - s_3) + s_3]^2 - i\epsilon} \quad (18)$$

with $s_1 - s_3 = p_3$ and $s_2 - s_3 = -p_2$. This integral can be solved in three steps [1]. First we integrate over x using a special transformation. This gives us three integrals of an

identical type over y . Next each of these is reduced to two simpler integrals over y and these are finally expressed in terms of dilogarithms (Spence functions). The first two steps define the quantities necessary to calculate the arguments of these dilogarithms, the last one is concerned with the problem of adding these dilogarithms. Of course the numerical problems may cause us to use a slightly different approach for the last two steps.

3.2 The arguments of the dilogarithms

The first step is accomplished by the transformation $y' = y + \alpha x$ with α a solution of $r \cdot r = 0$, $r = p_3 + \alpha p_2$:

$$\alpha = \frac{-p_2 \cdot p_3 + \sigma_\alpha \sqrt{-\Delta_2}}{p_2^2} \quad (19)$$

with

$$\Delta_2 = \delta_{p_i p_{i+1}}^{p_i p_{i+1}} = p_i^2 p_{i+1}^2 - (p_i \cdot p_{i+1})^2 \quad (\text{for any } i) \quad (20)$$

(The choice of $\sigma_\alpha = \pm 1$ is still free.) Note that the vertex must be part of a physical diagram, so that α must be real. (We do not encounter three spacelike momenta in scattering and decay experiments). We change the order of integration and integrate over x . This results in three integrals:

$$\begin{aligned} \frac{C_0}{i\pi^2} = & - \int_0^1 dy \frac{\log[(s_3 - yp_2)^2 - i\epsilon]}{2(s_3 \cdot r - p_2 \cdot ry)} \\ & + \int_0^\alpha dy \frac{\log[(s_3 + p_3 \frac{y}{\alpha})^2 - i\epsilon]}{2(s_3 \cdot r - p_2 \cdot ry)} \\ & + \int_\alpha^1 dy \frac{\log[(s_2 - p_1 \frac{1-y}{1-\alpha})^2 - i\epsilon]}{2(s_3 \cdot r - p_2 \cdot ry)} \end{aligned} \quad (21)$$

To obtain a more symmetric form we make the transformation $y \rightarrow \alpha(1-y)$ in the second integral and $y \rightarrow 1 - (1-\alpha)y$ in the third integral. This brings some order in the arguments of the logarithms. The denominators are all of the form $a(y - y_i)$ for $i = 1, 2, 3$. We can solve for the y_i by substituting the r and α in the denominators. These y_i turn out to be the root of very compact quadratic equations

$$p_i^2 y_i^2 - 2y_i p_i \cdot s_{i+1} + (\delta_{p_i p_{i+1}}^{p_i p_{i+1}})^2 / \Delta_2 = 0 \quad (22)$$

$$y_i = \frac{1}{p_i^2} \left(p_i \cdot s_{i+1} + \sigma_\alpha \delta_{p_i p_{i+1}}^{p_i p_{i+1}} / \sqrt{-\Delta_2} \right) \quad (23)$$

with

$$\left(\delta_{\mu s_{i+1}}^{p_i p_{i+1}} \right)^2 / \Delta_2 = s_{i+1}^2 - \delta_{s_1 s_2 s_3}^{s_1 s_2 s_3} / \Delta_2 \quad (24)$$

Finally we add extra terms with $y \rightarrow y_i$, which add up to zero, to cancel the residue of the pole at y_i [1]. The result is

$$\frac{C_0}{i\pi^2} = \frac{\sigma_\alpha}{2\sqrt{-\Delta_2}} \int_0^1 dy \sum_{i=1}^3 \frac{1}{y - y_i} \left(\begin{array}{l} \log[(s_{i+1} - yp_i)^2 - i\epsilon] \\ - \log[(s_{i+1} - y_i p_i)^2 - i\epsilon] \end{array} \right) \quad (25)$$

We have to choose the labeling of the external momenta such that p_2^2 is nonzero if we want to use the above formulae (see eq.19). But if one of the p_i^2 is zero it is smarter to call this momentum p_2 anyway and to not even bother about introducing α . We can integrate directly over y in equation (18). This case is sufficiently simple so that we will not consider it in the context of this paper.

From here on there are two ways to proceed, depending on how bad the numerical problems are. Proceeding normally one finds an expression with 12 dilogarithms. There is however one source of instability which is a result of the addition of two integrals to form the first integral in equation (21). If we do not combine those integrals we will have four integrals which may show cancellations in pairs. This happens for instance in the vertex correction to $ee\gamma$. The loop with two e 's and a Z_0 in it becomes problematic when p_γ^2 is small and spacelike. $p_\gamma^2 = -10^{-20}\text{GeV}^2$ gives a loss of 6 digits. This would be bearable if it were the only source of problems, which it isn't of course. In this case we have to keep all 16 dilogarithms. The choice of the α becomes rather important here: if $\alpha < 0$ or $\alpha > 1$ the integration may give rise to unnecessarily large similar contributions with opposite signs. All these numerical troubles will manifest themselves in the form of pairs or quartets of dilogarithms the sum of which is much smaller than the individual terms. This is discussed further in 3.3.4.

In the normal scheme the next step is to find the roots of the arguments of the logarithms in equation (25). These roots, called z_i^\pm , are defined by

$$p_i^2 z_i^2 - 2s_{i+1} p_i z + s_{i+1}^2 = 0 \quad (26)$$

$$z_i^\pm = \frac{p_i \cdot s_{i+1} \pm \sqrt{-\delta_{p_i s_{i+1}}^{p_i s_{i+1}}}}{p_i^2} \quad (27)$$

The special case $p_i^2 = 0$ is trivial, but should have been treated differently anyway as mentioned before. For stability in later equations we also need expressions for the overcomplete set $1 - y_i$, $1 - z_i^\pm$, $y_i - z_i^\pm$, $2y_i - z_i^- - z_i^+$ and $y_i - 2z_i^\pm$. These follow naturally from the definitions, but have to be rewritten so that their evaluation does not cause problems.

- $1 - y_i$ is the solution of $p_i^2(1 - y_i)^2 + 2(1 - y_i)p_i \cdot s_i + \left(\delta_{p_i p_{i+1}}^{\mu s_i}\right)^2 / \Delta_2 = 0$ with the sign $-\sigma_\alpha$. It is part of the argument of the dilogarithm (see equation (29)) and can be much smaller than y_i . For instance in the diagram $ee\gamma$ with two e 's and a γ with a small mass λ for regularization in the loop, $y_2 \approx 1 - (\lambda/m_e)^2$.
- $1 - z_i^\pm$ follows from $p_i^2(1 - z_i^\pm)^2 + 2(1 - z_i^\pm)p_i \cdot s_i + s_i^2 = 0$. This value is needed when cancellations among the dilogarithms have to be circumvented. It can be as small as $1 - y_i$.

- One of the $y_i - z_i^\pm$ is stable, the other one can be obtained accurately with

$$(y_i - z_i^+)(y_i - z_i^-) = \delta_{p_i p_{i+1} s_{i+1}}^{p_i p_{i+1} s_{i+1}} / (p_i^2 \Delta_2) \quad (28)$$

Note that the 3×3 determinant $\delta_{p_i p_{i+1} s_{i+1}}^{p_i p_{i+1} s_{i+1}} = \delta_{s_1 s_2 s_3}^{s_1 s_2 s_3}$ is unique. This quantity, if not calculated this way, would have given problems in many vertex corrections. In the same infra-red divergent diagram $ee\gamma$ with the two e 's and a γ in the loop, $(y_i - z_i^+) / (y_i + z_i^+)$ is of order $(\lambda/m_e)^4$.

- $2y_i - z_i^+ - z_i^- = 2\sigma_\alpha \delta_{p_i p_{i+1} s_{i+1}}^{p_i p_{i+1} s_{i+1}} / (p_i^2 \sqrt{-\Delta_2})$ and $y_i^+ - 2z_i^\pm = -y_i^- \mp 2\sqrt{-\delta_{p_i p_{i+1} s_{i+1}}^{p_i p_{i+1} s_{i+1}} / p_i^2}$. These combinations are needed in some rare cases.

It should be clear by now why we are using the generalised Kronecker deltas and the vectors s_i . All numerical problems can be concentrated in a very small number of these determinants, which are functions of physical quantities, and the formulae are very compact.

3.3 Cancellations among the dilogarithms

With the roots introduced in the previous section we can split up the logarithms in equation (25). If we assign $z_i^\pm \rightarrow z_i^\pm \mp i\epsilon$ we do not cross any branch cuts. This leads to the functions

$$\begin{aligned} R(y_i, z_i, \pm i\epsilon) &= \int_0^1 \frac{dy}{y - y_i} (\log(y - z_i \pm i\epsilon) - \log(y_i - z_i \pm i\epsilon)) \\ &= \text{Li}_2(c_1) - \text{Li}_2(c_2) + \eta_1 \log(c_1) - \eta_2 \log(c_2) \end{aligned} \quad (29)$$

with

$$c_1 = y_i / (y_i - z_i \pm i\epsilon) \quad (30)$$

$$c_2 = (y_i - 1) / (y_i - z_i \pm i\epsilon) \quad (31)$$

$$\eta_1 = \eta(-z_i \pm i\epsilon, 1 / (y_i - z_i \pm i\epsilon)) \quad (32)$$

$$\eta_2 = \eta(1 - z_i \pm i\epsilon, 1 / (y_i - z_i \pm i\epsilon)) \quad (33)$$

The dilogarithm Li_2 is defined by

$$\text{Li}_2(z) = - \int_0^z \frac{\log(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (34)$$

(the last equality holds for $|z| < 1$ only) and the function η compensates for the cut in the Riemann sheet of the logarithm:

$$\log(ab) = \log(a) + \log(b) + \eta(a, b) \quad (35)$$

If the parameters y_i, z_i are real the η -terms vanish in equation (29).

The dilogarithms we obtain by the above methods all have the same overall factor. The ways we obtain these dilogarithms make it possible that we add big contributions to some integrals and subtract them from other integrals. These big contributions will show up in the dilogarithms and the logarithms. It is therefore quite normal that there will be dilogarithms of which the values are nearly equal but the sign is opposite. This is the case when the arguments of those dilogarithms are close to each other, which in turn often occurs when the overall factor $1/\sqrt{-\Delta_2}$ is large. In an s-channel reaction this is the case when there isn't much phase space for the reaction. In a t-channel reaction this is the case when the scattering is in the forward region which happens very frequently.

For such a pair of dilogarithms we can use some identities [11] that leave us with a formula in which no individual term is much larger in size than the sum of the terms.

3.3.1 Transform the arguments

The first step is to transform the arguments of the dilogarithms to a region near zero. If $|x|, |1-x| > 1$ we use

$$\text{Li}_2\left(\frac{1}{x}\right) = -\text{Li}_2(x) - \frac{1}{2}\log^2(-x) - \frac{\pi^2}{6} \quad (36)$$

to map the arguments c_i to $c'_i = 1/c_i$ near 0. In case that $\text{Re}(x) > 1/2, |1-x| < 1$ we use

$$\text{Li}_2(1-x) = -\text{Li}_2(x) - \log(x)\log(1-x) + \frac{\pi^2}{6} \quad (37)$$

to map to new arguments $c'_i = 1 - c_i$, which can be calculated accurately if z and $1-z$ are known. This transformation also circumvents many problems associated with the singularity in the derivative of $\text{Li}_2(x)$ for $x = 1$. The terms $\pi^2/6$ should be kept apart as they often cancel each other. This is best done by keeping the multiple of $\pi^2/12$ in a separate integer. If there is anything left in the end it can be added to the rest. This avoids spoiling the accuracy of the intermediate results.

An illustrating example is the scalar three-point function occurring in the $\nu\nu\gamma$ vertex with two e 's and a W in the loop and p_γ^2 small and spacelike. In this case the value $p_\gamma^2 = -10^{-8}\text{GeV}^2$ gives $y_1 \approx -m_W^2/(2m_e\sqrt{-p_\gamma^2}) \approx -6\cdot 10^{11}$, $z_1^- \approx m_e/\sqrt{-p_\gamma^2} \approx 5$. We see that the arguments of the dilogarithms c_1 and c_2 in equation (29) are both very close to 1, however, $c'_1 = 1 - c_1$ and $c'_2 = 1 - c_2$ are of order 10^{-12} , which is also the order of magnitude of all other dilogarithms.

3.3.2 Arguments close together

Next we notice that within this region there are cancellations if the transformed arguments are close together, $c'_1 \approx c'_2$. First we consider the dilogarithms, next the logarithms resulting from the transformation or the η terms. Two cases have to be distinguished:

1. If the (transformed) arguments are close together but not small we use the Hill identity¹ in the form:

$$\begin{aligned} \text{Li}_2(ab) &= \text{Li}_2(a) + \text{Li}_2(b) - \text{Li}_2\left(\frac{a(1-b)}{1-ab}\right) - \text{Li}_2\left(\frac{b(1-a)}{1-ab}\right) \\ &\quad - \log\left(\frac{1-a}{1-ab}\right) \log\left(\frac{1-b}{1-ab}\right) \\ &\quad - \eta(1-a, 1/(1-ab)) \log(a) - \eta(1-b, 1/(1-ab)) \log(b) \end{aligned} \quad (38)$$

with $a = c'_1$ and $b = c'_2/c'_1 \approx 1$. We then obtain

$$\begin{aligned} \text{Li}_2(c'_1) - \text{Li}_2(c'_2) &= \text{Li}_2(1 - c'_2/c'_1) + \text{Li}_2\left(\frac{c'_1(1 - c'_2/c'_1)}{1 - c'_2}\right) - \text{Li}_2\left(\frac{1 - c'_2/c'_1}{1 - c'_2}\right) \\ &\quad + \log(c'_2/c'_1) \log(1 - c'_2) \end{aligned} \quad (39)$$

Some of the arguments of these dilogarithms have a very simple form when expressed in the original parameters y and z . These are shown for each of the three possible transformations:

$$\left\{ \begin{array}{l} c'_i = c_i : \quad 1 - c'_2/c'_1 = \frac{1}{y} \quad \frac{c'_1(1 - c'_2/c'_1)}{1 - c'_2} = \frac{1}{1-z} \\ c'_i = 1/c_i : \quad 1 - c'_2/c'_1 = \frac{1}{1-y} \quad \frac{1 - c'_2/c'_1}{1 - c'_2} = \frac{1}{1-z} \\ c'_i = 1 - c_i : \quad 1 - c'_2/c'_1 = \frac{1}{z} \quad \frac{c'_1(1 - c'_2/c'_1)}{1 - c'_2} = \frac{1}{1-z} \end{array} \right. \quad (40)$$

2. In case that $c'_i \ll 1$ the result is of order $c'_1(1 - c'_2/c'_1)$, but the terms in (39) are only of order $(1 - c'_2/c'_1)$. However, a Taylor expansion is feasible in this region:

$$\text{Li}_2(c'_1) - \text{Li}_2(c'_2) = \sum_{n=1}^{\infty} \frac{c'_1{}^n}{n^2} \left(1 - \left(\frac{c'_2}{c'_1}\right)^n\right) \quad (41)$$

with $\{1 - (c'_2/c'_1)^n\} = \{1 - (c'_2/c'_1)^{n-1}\} + (1 - c'_2/c'_1)(c'_2/c'_1)^{n-1}$ easily calculated.

The logarithms can be similarly rewritten. If there has been no transformation the expression

$$\eta_1 \log(c_1) - \eta_2 \log(c_2) = \eta_2 \log(c_2/c_1) + (\eta_1 - \eta_2) \log(c_1) \quad (42)$$

is stable. In case we transformed $c'_i = 1/c_i$ we use

$$\begin{aligned} \eta_1 \log(c_1) - \eta_2 \log(c_2) &- \frac{1}{2} \log^2(-c_1) + \frac{1}{2} \log^2(-c_2) \\ &= \log(c'_2/c'_1) \left(\eta_2 + \log(-c'_2) - \frac{1}{2} \log(c'_2/c'_1) \right) + (\eta_2 - \eta_1) \log(-c'_1) \end{aligned} \quad (43)$$

¹the η -terms arise from the possibility that a , b and ab are not on the same Riemann sheet. They are often not mentioned in the mathematical literature.

The transformation and η -terms for $c'_i = 1 - c_i$ become

$$\begin{aligned} \eta_1 \log(c_1) - \eta_2 \log(c_2) - \log(c_1) \log(1 - c_1) - \log(c_2) \log(1 - c_2) \\ = \log(c_1) (\log(c'_2/c'_1) + \eta_1 - \eta_2) + (\log(c'_2) + \eta_2) \log(c_2/c_1) \end{aligned} \quad (44)$$

with $c_2/c_1 = 1 - 1/y$. In all cases it is assumed that the small logarithm $\log(c'_2/c'_1)$ does not have an associated η term, this should be checked for.

The importance of the cancellations between the dilogarithms can be illustrated in the same $\nu\nu\gamma$ diagram. If we give the neutrino a small but finite mass we have $y_3 \approx z_3^- \approx -m_W^2/m_\nu^2$, $y_3 - z_3^- = -m_W^2/p_\gamma^2$. The original arguments $c_{1,2}$ are thus very large (about p_γ^2/m_ν^2), but very nearly equal: the sum of these dilogarithms is proportional to $m_\nu^2 m_e/m_W^3$, which of course tends to zero as $m_\nu \rightarrow 0$.

3.3.3 Special cases

Finally there are a few special cases, mainly for use in transformed four-point functions. In case an argument c_i is close to -1 (but not close to the other argument) we use the Hill identity with $x = c_i$, $y = -c_i^{-1}$ to obtain

$$\begin{aligned} \text{Li}_2(x) = \text{Li}_2(-1) + \text{Li}_2\left(\frac{2y - z}{y - z}\right) - \text{Li}_2\left(-\frac{2y - z}{y - z}\right) \\ - \text{Li}_2(2y - z) + \log\left(\frac{1}{y - z}\right) \log(1 - 2y + z) \end{aligned} \quad (45)$$

In this equation $\text{Li}_2(-1) = -\pi^2/12$, two of these terms may cancel a $\pi^2/6$ from the transformations. The argument $2y - z$ is obtained from $2y - z^+ - z^-$, which is known.

Another problem may occur if the roots z_i^\pm are complex. As they are each others complex conjugate and all functions are analytic we only need the real part of the dilogarithms, which may be much smaller than the absolute value. In this case the evaluation of the complex dilogarithm should be done with great care.

A similar condition comes up if both roots are (almost) equal and the arguments of the dilogarithms close to 2. In this case we also do not use the imaginary part. The real part is given by

$$\text{Li}_2(2 - x) = \pi^2/4 - x^2/4 - x^3/6 - 5x^4/48 - x^5/15 - \dots \quad (46)$$

with $2 - c_1 = (y - 2z)/(y - z)$, $c_2 = ((1 - y) - 2(1 - z))/(y - z)$ known accurately.

Finally, there is one region in which there are cancellations between two R functions. This occurs if $|z_i^\pm| \gg |y_i|$, for instance in the vertex correction to $\gamma^* \gamma^* X$, X light, with three W 's in the loop. Then

$$\text{Li}_2\left(\frac{y_i}{y_i - z_i^+}\right) + \text{Li}_2\left(\frac{y_i}{y_i - z_i^-}\right) \quad (47)$$

is almost of the type

$$\text{Li}_2(x) + \text{Li}_2(-x) = \frac{1}{2}\text{Li}_2(x^2) \quad |x| < 1 \quad (48)$$

We thus transform the second dilogarithm with this identity and treat the resulting cancellation with the Hill identity to obtain

$$\begin{aligned} R(y, z^+) + R(y, z^-) &= R\left(Y, \frac{z^+}{y}Y\right) - R\left(Y, \frac{1-z^+}{1-y}Y\right) \\ &+ \frac{1}{2}\text{Li}_2\left(\left(\frac{y}{y-z^-}\right)^2\right) - \frac{1}{2}\text{Li}_2\left(\left(\frac{y-1}{y-z^-}\right)^2\right) \end{aligned} \quad (49)$$

with $Y = (y - z^-)/(2y - z^+ - z^-)$.

3.3.4 16 dilogarithms

In case we need to use the formulation with 16 dilogarithms exactly the same methods are used to circumvent the cancellations between two R functions.

$$\begin{aligned} \frac{C_0}{i\pi^2} &= \frac{1}{2\sqrt{-\Delta_2}} \left(R(y_1, z_1^+) - R(y_1, w_1^-) + R(y_1, z_1^-) - R(y_1, w_1^+) \right. \\ &\quad \left. + (w_1, y_1, z_1 \rightarrow w_3, y_3, z_3) \right) \end{aligned} \quad (50)$$

with $w_1^\pm = (1 - z_2^\pm)/(1 - \alpha)$, $w_3^\pm = z_2^\pm/\alpha$, The differences $(w_i^\pm - z_i^\mp)$ can be calculated accurately for special cases. The dilogarithms cancel in groups of four. They can be combined differently if there are only pairwise cancellations, otherwise the Hill identity or a Taylor expansion are used:

$$\begin{aligned} R(y, z) - R(y, w) &= R\left(\frac{w-y}{w-z}, \frac{z w - y}{y w - z}\right) - R\left(\frac{w-y}{w-z}, \frac{1-z w - y}{1-y w - z}\right) \end{aligned} \quad (51)$$

$$= R\left(\frac{1-w}{y-w}, z \frac{1-w}{z-w}\right) - R\left(1-w, z \frac{1-w}{z-w}\right) - \log\left(\frac{w-1}{w}\right) \log\left(\frac{w-z}{w-y}\right) \quad (52)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{y}{y-z}\right)^n \left(1 - \left(1 - \frac{1}{y}\right)^n\right) \left(1 - \left(\frac{y-z}{y-w}\right)^n\right) \quad (53)$$

3.3.5 Conclusion

The expressions needed for the (new) arguments of the dilogarithms and logarithms can all be calculated accurately if one knows accurate values for the overcomplete set $y_i, z_i^\pm, 1 - y_i, 1 - z_i^\pm, y_i - z_i^\pm, 2y_i - z_i^+ - z_i^-$ and $y_i - 2z_i^\pm$

3.4 Complex arguments

The arguments of the three-point function may be complex. This occurs when the finite width of the internal particles is taken into account or when α is complex due to an

unusual kinematic configuration. This last configuration may occur when calculating three-point functions with transformed momenta for the four-point function (see section 4). In case of complex parameters some complications may arise. These stem from the careful bookkeeping we have to perform to keep track of the Riemann sheets.

The main problems occur when splitting up

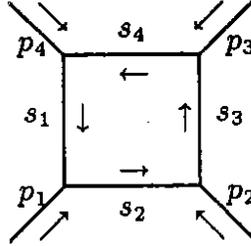
$$\log[(s_{i+1} - yp_i)^2 - i\epsilon] = \log(p^2) + \log(y - z_i^+ - i\epsilon) + \log(y - z_i^- + i\epsilon) + \eta((y - z_i^+ - i\epsilon), (y - z_i^- + i\epsilon)) \quad (54)$$

This last term gives rise to additions $2\pi n \log(1 - 1/y_i)$ to the sum of dilogarithms.

The treatment of the dilogarithms is analogous to the discussion in section 3.3, except that now the η terms cannot be ignored. The cancellation between the imaginary parts of many terms is no longer exact and has to be calculated numerically. Great care has to be taken to distinguish when the complex part of the argument is really zero, so that the $i\epsilon$ prescription has to be followed, and when it is just very small.

4 The four-point function

The scalar four-point function is represented by the diagram



in which we have introduced the vectors s_i again. The corresponding integral is

$$\begin{aligned} & D_0(m_1^2, m_2^2, m_3^2, m_4^2, p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2) \\ &= \int \frac{d^4 Q}{(Q^2 - m_1^2)((Q + p_1)^2 - m_2^2)((Q + p_1 + p_2)^2 - m_3^2)((Q + p_1 + p_2 + p_3)^2 - m_4^2)} \\ &= i\pi^2 \int dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) \theta(x_1) \theta(x_2) \theta(x_3) \theta(x_4) \\ & \quad \times \{[x_1 s_1 + x_2 s_2 + x_3 s_3 + x_4 s_4]^2 - i\epsilon\}^{-2} \end{aligned} \quad (55)$$

$$\begin{aligned} &= i\pi^2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \\ & \quad \times \{[x_1(s_1 - s_4) + x_2(s_2 - s_4) + x_3(s_3 - s_4) + s_4]^2 - i\epsilon\}^{-2} \end{aligned} \quad (56)$$

In the above formula $s_1 - s_4 = p_4$, $s_2 - s_4 = -p_2 - p_3$ and $s_3 - s_4 = -p_3$.

If we were to use a transformation of the type that introduced α for the three-point function things would become rather complicated. We would still have two integration

parameters left and we would have to use the same trick once more on each of the terms in these integrals. Luckily there exists a more elegant transformation that can be used in nearly all cases [1]. After this transformation has been applied one integration becomes trivial. Once that integration has been done we are left with the equivalent of two three-point functions.

4.1 The transformation

We use the transformation

$$x_i = \frac{A_i x'_i}{\sum_{j=1}^4 A_j x'_j} \equiv \frac{A_i x'_i}{N} \quad (57)$$

with $x'_4 = 1 - x'_1 - x'_2 - x'_3$, no summation over i implied. The Jacobian of this transformation is

$$J = |A_1 A_2 A_3 A_4 N^{-4}| \quad (58)$$

This transformation keeps the boundaries fixed, although the integration region changes from inside $\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)$ to some region outside if one or more of the A_i is negative. For the moment we only consider the case $A_i > 0$. In anticipation to the full result we drop the absolute value signs.

Applying this transformation (57) to (55) we see that, apart from an overall factor $A_1 A_2 A_3 A_4$ it is equivalent to the substitution $s'_i = A_i s_i$. We like to choose the A_i such that

$$(s'_3 - s'_4)^2 = 0 \quad (59)$$

$$(s'_2 - s'_4) \cdot (s'_3 - s'_4) = 0 \quad (60)$$

$$(s'_1 - s'_4) \cdot (s'_3 - s'_4) = 0 \quad (61)$$

This makes the integral over x'_3 trivial. There is however one condition. The transformation can only give rise to physical vectors if all A_i are real. So let us have a look at the conditions that the A_i have to fulfill. We will express everything in terms of A_4 , which is only a scale factor. In that case the equation for A_3 is a plain quadratic equation and the other equations give a linear expression that expresses the other A_i in terms of A_3 and A_4 . Thus the solution is real if $\delta_{s_3 s_4}^{s_3 s_4} \leq 0$. This implies that the vertex (s_3, s_4, p_3) corresponds to a physical process. We could have used any permutation of the vectors s_i when choosing which one is s_3 and which one is s_4 . Four permutations correspond to a choice of one of the four vertices, the other two correspond to the s and the t channel. It is possible to find diagrams for which all six δ 's are positive, but $t < 0$ or $s > (m_1 + m_3)^2$ are already sufficient to obtain a real solution. In case no vertex is physical the projective transformation cannot be used in its current form. We will not consider that case here and from now on we assume that the s_i have been permuted in such a way that all A_i are real.

If $\delta_{s_3 s_4}^{s_3 s_4} = 0$ there are some problems in the common case that $s_3^2 = s_4^2, p_3^2 = 0$, as then $A_1 = A_2 = 0$. By regularizing $m_4 = m_3(1 + \delta)$ this limit can be taken.

The equations for the A_i in the form (59)-(61) are not suited for numerical evaluation. Instead we use

$$A_i^{-2} s_4^2 - 2A_i^{-1} A_4^{-1} s_i \cdot s_4 + A_4^{-2} \left(\delta_{s_3 s_4}^{s_i \mu} \right)^2 / \delta_{s_3 s_4}^{s_3 s_4} = 0 \quad (62)$$

$$A_i^{-1} = \frac{s_i \cdot s_4 + \sigma_3 \delta_{s_3 s_4}^{s_i s_4} / \sqrt{-\delta_{s_3 s_4}^{s_3 s_4}}}{s_4^2} A_4^{-1} \quad (63)$$

The variable σ_3 takes the values ± 1 .

The new vectors s'_i define new vectors $p'_{ij} = s'_i - s'_j$. The transformed dotproducts ($s'_i \cdot s'_j, p'_{ij} \cdot s'_k$ and $p'_{ij} \cdot p'_{kl}$) can all be expressed as solutions of quadratic equations having determinants as coefficients, with the exception of some of the $p'_{ij} \cdot p'_{kl}$.

$$s'_i \cdot s'_j = A_i A_j s_i \cdot s_j \quad (64)$$

$$s'_i \cdot p'_{jk} = s'_i \cdot (s'_j - s'_k) = A_i A_j A_k B_{ijk} \quad (65)$$

with B_{ijk} a solution of a quadratic equation

$$s_4^2 B_{ijk}^2 - 2\delta_{s_j s_k}^{s_i s_4} B_{ijk} + \left(\delta_{s_j s_k}^{s_i \mu} \delta_{s_3 s_4}^{\mu \nu} \right)^2 / \delta_{s_3 s_4}^{s_3 s_4} = 0 \quad (66)$$

$$B_{ijk} = \frac{\delta_{s_j s_k}^{s_i s_4} - \sigma_3 \delta_{s_j s_k}^{s_i \mu} \delta_{s_3 s_4}^{\mu s_4} / \sqrt{-\delta_{s_3 s_4}^{s_3 s_4}}}{s_4^2} \quad (67)$$

There are of course only six non-trivial dotproducts $p'_{ij} \cdot p'_{lk}$. They are given by

$$p'_{ij} \cdot p'_{kl} = (s'_i - s'_j) \cdot (s'_k - s'_l) = A_i A_j A_k A_l C_{ijkl} \quad (68)$$

$$C_{ijkl} = \frac{1}{s_4^4} \left\{ \delta_{s_i s_j}^{\mu \nu} \delta_{s_k s_l}^{\mu \rho} \left(s_4^\nu s_4^\rho \delta_{s_3 s_4}^{s_3 s_4} - \delta_{s_3 s_4}^{\nu s_4} \delta_{s_3 s_4}^{\rho s_4} \right) / \delta_{s_3 s_4}^{s_3 s_4} \right. \\ \left. + \sigma_3 \delta_{s_i s_j}^{\mu \nu} \delta_{s_k s_l}^{\mu \rho} \left(s_4^\nu \delta_{s_3 s_4}^{\rho s_4} + s_4^\rho \delta_{s_3 s_4}^{\nu s_4} \right) / \sqrt{-\delta_{s_3 s_4}^{s_3 s_4}} \right\} \quad (69)$$

The equations for $p'_{14}{}^2$ and $p'_{24}{}^2$ simplify to the form

$$p'_{i4} \cdot p'_{k4} = A_i A_k A_4^2 \delta_{s_k s_3 s_4}^{s_i s_3 s_4} / \delta_{s_3 s_4}^{s_3 s_4} \quad (70)$$

This formula is also correct for $p'_{14} \cdot p'_{24}$. The choice of the root for A_3 is important to keep the expressions for the general case in a stable form.

4.1.1 Reduction to three-point functions

In case all $A_i > 0$ we integrate over x'_3 . This gives two terms each of which has the form of a three-point function integral.

$$\frac{D_0}{i\pi^2} = \frac{A_1 A_2 A_3 A_4}{s_3'^2 - s_4'^2} \{C_0(s'_1, s'_2, s'_3) - C_0(s'_1, s'_2, s'_4)\} \quad (71)$$

As we know all transformed dotproducts we can evaluate these three-point functions with the methods described in section 3. The numerical problems are in general a lot worse than in the case of three-point functions with physical parameters, but mostly manageable in our algorithms.

Cancellations between the three-point functions occur when $|s_3'^2 - s_4'^2| \ll s_3'^2, s_4'^2$. Because

$$s_3'^2 - s_4'^2 = 2\sigma_3 A_3 A_4 \sqrt{-\delta_{s_3 s_4}^{s_3 s_4}} \quad (72)$$

this happens when the (s_3, s_4, p_3) vertex is near a (pseudo) threshold, $|\delta_{s_3 s_4}^{s_3 s_4}| \ll s_3^2 s_4^2$. Common situations are $s \approx (m_1 + m_3)^2$ or $t \approx (m_2 - m_4)^2$. To accurately calculate the difference of these three-point functions we transform again as in section 3.3. The small parameters now are

$$1 - c_1^{(4)}/c_1^{(3)} \propto y_i^{(3)} z_i^{(4)} - y_i^{(4)} z_i^{(3)} \quad (73)$$

$$1 - c_2^{(4)}/c_2^{(3)} \propto (1 - y_i^{(3)})(1 - z_i^{(4)}) - (1 - y_i^{(4)})(1 - z_i^{(3)}) \quad (74)$$

The superscript refers to the s_i that is included in the three-point function. These small parameters can be calculated in a straightforward manner from the definitions (23) and (27):

$$y_1^{(3)} z_1^{\pm(4)} - y_1^{(4)} z_1^{\pm(3)} = -\sigma_\alpha \frac{p_3' \cdot s_2'}{\sqrt{-\Delta_2'}} z_1^{\pm} \quad (75)$$

$$\begin{aligned} y_2^{(3)} z_2^{\pm(4)} - y_2^{(4)} z_2^{\pm(3)} &= \frac{1}{p_2'^2} \left\{ \sigma_\alpha \left(p_2'^2 p_3' \cdot s_4' p_1' \cdot s_4' \mp p_1' \cdot p_2' p_3' \cdot s_4' \sqrt{-\delta_{23}} \right. \right. \\ &\quad \left. \pm \delta_{p_2' p_3'}^{p_2' s_3'} (\sqrt{-\delta_{23}} - \sqrt{-\delta_{24}}) \right) / \sqrt{-\Delta_2'} \\ &\quad \left. \pm \left(p_3' \cdot s_4' \sqrt{-\delta_{23}} + p_2' \cdot s_4' (\sqrt{-\delta_{23}} - \sqrt{-\delta_{24}}) \right) \right\} \quad (76) \end{aligned}$$

$$\begin{aligned} y_3^{(3)} z_3^{\pm(4)} - y_3^{(4)} z_3^{\pm(3)} &= \frac{1}{p_4'^2} \left\{ \sigma_\alpha \left(p_4'^2 p_3' \cdot s_1' p_1' \cdot s_1' \mp p_1' \cdot p_4' p_3' \cdot s_1' \sqrt{-\delta_{31}} \right. \right. \\ &\quad \left. \pm \delta_{p_3' p_1'}^{p_3' s_1'} (\sqrt{-\delta_{31}} - \sqrt{-\delta_{41}}) \right) / \sqrt{-\Delta_2'} \\ &\quad \left. \pm \left(p_3' \cdot s_1' \sqrt{-\delta_{31}} + p_{31}' \cdot s_1' (\sqrt{-\delta_{31}} - \sqrt{-\delta_{41}}) \right) \right\} \quad (77) \end{aligned}$$

The dilogarithms are rewritten as in section 3.3.4:

$$R(y, z)^{(3)} - R(y, z)^{(4)} = R(Ay, Az) - R(B(1-y), B(1-z)) \quad (78)$$

with

$$A = \frac{y^{(4)} - z^{(4)}}{y^{(3)}z^{(4)} - y^{(4)}z^{(3)}} \quad (79)$$

$$B = \frac{z^{(4)} - y^{(4)}}{(1 - y^{(3)})(1 - z^{(4)}) - (1 - y^{(4)})(1 - z^{(3)})} \quad (80)$$

4.1.2 The overall factor

The Δ'_2 , which is the same for both three-point functions, can be reduced to the 4×4 determinant of the untransformed parameters

$$\begin{aligned} \Delta'_2 &= \delta_{p'_{14} p'_{24}} \\ &= A_1^2 A_2^2 \left(\delta_{s_1 s_3 s_4} \delta_{s_2 s_3 s_4} - (\delta_{s_2 s_3 s_4})^2 \right) / (\delta_{s_3 s_4})^2 \\ &= A_1^2 A_2^2 \delta_{s_1 s_2 s_3 s_4} / \delta_{s_3 s_4} \end{aligned} \quad (81)$$

The complete overall factor of the dilogarithms is thus

$$\frac{A_1 A_2 A_3 A_4 \sigma_\alpha}{s_3'^2 - s_4'^2} \frac{1}{2\sqrt{-\Delta'_2}} = \frac{\sigma_\alpha \sigma_3 \epsilon(A_1) \epsilon(A_2)}{4\sqrt{\delta_{s_1 s_2 s_3 s_4}}} \quad (82)$$

($\epsilon(x) = \pm 1$ depending on the sign of x).

We can see that if at least one of the six three-point vertices is unphysical (which will most often be the case) the s_i have an imaginary component, thus $\delta_{s_1 s_2 s_3 s_4} = -(\epsilon^{s_1 s_2 s_3 s_4})^2 > 0$. In that case $\Delta_2 < 0$ and the transformed three-point functions are physical. However, there are diagrams in which the s_i are real so that we have to calculate three-point functions with a complex α .

In general, cancellations among the dilogarithms will occur when the overall factor is large, i.e. $\epsilon^{s_1 s_2 s_3 s_4}$ is small. Contrary to the overall factor of the three-point function, this overall factor does not have a simple kinematical interpretation, although it is of course also zero when three p_i are linearly dependent, for instance when scattering in the forward region.

4.1.3 Extra terms

If one or more of the A_i are negative the integration region of the corresponding x_i changes from $0 < x_i < 1$ to $(x_i < 0) \cup (x_i > 1)$. Adding and subtracting the integral from $-\infty$ to $+\infty$ one regains the original integral without the absolute value on the Jacobian (71) plus some extra terms, which arise from this (simpler) integral. They are given (up to the overall factor) by

$$R_{ij} = \pm \theta(-A_i A_j) i\pi \log \left(-\frac{y'_k - z_k^-}{y'_k - z_k^+} \right) \quad (83)$$

with k the integral in which both s'_i and s'_j appear and the minus sign is taken when this integral is part of $C_0(s'_1, s'_2, s'_4)$.

There is one extra term which appears when $A_3A_4 < 0$ because of the interplay between the pole in x'_3 at infinity after the projective transformation and the integration region which reaches out to there.

$$S = \theta(-A_3A_4) \{ \epsilon(A_1A_2)(S_{12} + S_{14}) - 2\theta(-A_1A_2)S_{24} \} \quad (84)$$

with

$$S_{ij} = -i\pi \log \left(-\frac{p'_i \cdot p'_j - \sqrt{-\Delta'_2}}{p'_i \cdot p'_j + \sqrt{-\Delta'_2}} \right) \quad (85)$$

A derivation is given in [1].

5 Tensor integrals

In realistic calculations we are rarely confronted with scalar loop integrals only. Many terms may have powers of the loop momentum in the numerator. Such integrals are called tensor integrals. These can be reduced to scalar integrals but the reduction scheme is rather involved. One way to do this is given in [2] and it has been programmed by M. Veltman into the program FormF. An algebraic version of this reduction scheme has been implemented in Reduce by R. Stuart [7]. This method of reduction suffers from some shortcomings though. First of all it is best suited for numerical implementation. This means that the numerical value of the scalar loop integrals is to be used to derive numerical values for the tensor integrals. These are expressed as tensors with numerical coefficients. In addition the results are not very accurate if the evaluation of kinematical determinants suffers from numerical instabilities when expanded in terms of dotproducts.

Here we present an algorithm that is better suited for an analytic reduction of formulae in terms of scalar loop integrals. In addition the instabilities have been concentrated in a number of determinants so that the formulae are numerically stable, as long as there are no cancellations among the scalar loop integrals. We will start with the four-point function tensor integrals in four dimensions as — amazingly enough — they are easiest. Next we discuss the extensions necessary to cover n dimensions and the three- and two-point integrals.

5.1 Four-point functions in four dimensions

Before starting the real work we introduce some new notations:

$$\underline{Q}^\mu = \int d^4Q \frac{Q^\mu}{N_1 N_2 N_3 N_4} \quad (86)$$

with

$$\begin{aligned}
N_1 &= Q^2 - m_1^2 \\
N_2 &= (Q + p_1)^2 - m_2^2 \\
N_3 &= (Q + p_1 + p_2)^2 - m_3^2 \\
N_4 &= (Q + p_1 + p_2 + p_3)^2 - m_4^2
\end{aligned} \tag{87}$$

Similarly we have

$$\underline{Q^\mu Q^\nu} = \int d^4 Q \frac{Q^\mu Q^\nu}{N_1 N_2 N_3 N_4} \tag{88}$$

and the integrals with higher powers of Q are written analogously. In addition we use some of the quantities of the previous chapters and

$$\Delta_3 = \delta_{p_1 p_2 p_3}^{p_1 p_2 p_3} \tag{89}$$

We may also be rather sloppy with the uncontracted upper and lower indices as eventually all indices are properly contracted so their nature as upper or lower indices is unimportant. The final result will be identical for Pauli metric and for Bjorken and Drell metric.

We start with the expansion of the determinant

$$\delta_{p_1 p_2 p_3 Q}^{p_1 p_2 p_3 \mu} = Q^\mu \Delta_3 - p_1^\mu \delta_{p_1 p_2 p_3}^{Q p_2 p_3} - p_2^\mu \delta_{p_1 p_2 p_3}^{p_1 Q p_3} - p_3^\mu \delta_{p_1 p_2 p_3}^{p_1 p_2 Q} \tag{90}$$

which gives the following decomposition for Q^μ

$$Q^\mu = \mathcal{P}^\mu + v^\mu v_\alpha Q^\alpha / \Delta_3 \tag{91}$$

with

$$\begin{aligned}
\mathcal{P}^\mu &= \frac{1}{2} \delta_{p_1 p_2 p_3}^{\mu \alpha \beta} \delta_{p_1 p_2 p_3}^{Q \alpha \beta} / \Delta_3 \\
v^\mu &= \varepsilon^{p_1 p_2 p_3 \mu}
\end{aligned} \tag{92}$$

From the last definition it follows that

$$v^\mu v_\mu = \Delta_3 \tag{93}$$

The first integral becomes

$$\underline{Q^\mu} = \underline{\mathcal{P}^\mu} + \underline{v^\mu v_\alpha Q^\alpha} / \Delta_3 \tag{94}$$

From the definition of v it is rather easy to see that the integral $\underline{v_\alpha Q^\alpha}$ is zero [12]. The integral $\underline{Q^\mu}$ is after all a linear combination of the p_i^μ . For two powers of Q we obtain

$$\underline{Q^\mu Q^\nu} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu} + \underline{(\mathcal{P}^\mu v^\nu v_\alpha Q^\alpha + \mathcal{P}^\nu v^\mu v_\alpha Q^\alpha)} / \Delta_3 + v^\mu v_\nu \underline{v^\alpha v_\beta Q_\alpha Q^\beta} / \Delta_3^2 \tag{95}$$

Again a study of the possible tensor structure reveals that the terms linear in $v \cdot Q$ be zero. The use of $v^\mu v^\nu$ is a deviation from [2] that pays back handsomely (this reference uses $g^{\mu\nu}$ to define their structure functions). We contract the indices μ and ν to fix the $(v \cdot Q)^2$ integral:

$$\underline{Q^2} = \underline{\mathcal{P}^2} + \underline{v^\alpha v_\beta Q_\alpha Q^\beta} / \Delta_3 \quad (96)$$

It is easy to express Q^2 in terms of denominators ($Q^2 = N_1 + m_1^2$) but we will present some more useful relations shortly. The final expression for the two powers of Q is therefore

$$\underline{Q^\mu Q^\nu} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu} + \underline{v^\mu v_\nu (Q^2 - \mathcal{P}^2)} / \Delta_3 \quad (97)$$

The cubic and quartic expressions are derived analogously and yield the equations:

$$\underline{Q^\mu Q^\nu Q^\rho} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu \mathcal{P}^\rho} + \underline{(v^\mu v_\nu \mathcal{P}^\rho + v^\mu v_\rho \mathcal{P}^\nu + v^\nu v_\rho \mathcal{P}^\mu) (Q^2 - \mathcal{P}^2)} / \Delta_3 \quad (98)$$

(again the odd powers of $v \cdot Q$ give zero trivially).

$$\begin{aligned} \underline{Q^\mu Q^\nu Q^\rho Q^\sigma} &= \underline{\mathcal{P}^\mu \mathcal{P}^\nu \mathcal{P}^\rho \mathcal{P}^\sigma} \\ &+ \underline{(v^\mu v_\nu \mathcal{P}^\rho \mathcal{P}^\sigma + v^\mu v_\rho \mathcal{P}^\nu \mathcal{P}^\sigma + v^\mu v_\sigma \mathcal{P}^\nu \mathcal{P}^\rho + v^\nu v_\rho \mathcal{P}^\mu \mathcal{P}^\sigma + v^\nu v_\sigma \mathcal{P}^\mu \mathcal{P}^\rho + v^\rho v_\sigma \mathcal{P}^\mu \mathcal{P}^\nu)} \\ &\underline{(Q^2 - \mathcal{P}^2)} / \Delta_3 + \underline{v^\mu v_\nu v^\rho v_\sigma (Q^2 - \mathcal{P}^2)^2} / \Delta_3^2 \end{aligned} \quad (99)$$

To obtain the integrals with $v \cdot Q$ one has to consider also the integrals with $Q^2 Q^\mu$ and $Q^2 Q^\mu Q^\nu$ respectively without expanding the Q^2 . These are compared to the cubic and quartic expressions with one contraction. The last term in the quartic expressions is found after contracting μ with ν and ρ with σ . This gives $\underline{(v \cdot Q)^4} = \underline{(Q^2 - \mathcal{P}^2)^2} \Delta_3^2$.

In principle all terms in the right hand sides of the above rewrites of the tensor integrals can be expanded easily in terms of whole denominators, so we obtain either scalar four point integrals or tensor integrals with fewer denominators. In practice this can be done in several ways of which we show one. Here we worry only about \mathcal{P}^μ , the treatment of the $(Q^2 - \mathcal{P}^2)$ terms is done in the appendix. We rewrite:

$$\Delta_3 \mathcal{P}^\mu = Q \cdot p_1 \delta_{p_1 p_2 p_3}^{\mu p_2 p_3} + Q \cdot p_2 \delta_{p_1 p_2 p_3}^{p_1 \mu p_3} + Q \cdot p_3 \delta_{p_1 p_2 p_3}^{p_1 p_2 \mu} \quad (100)$$

Now we replace $Q \cdot p_i$ by $\frac{1}{2} N_{i+1} - \frac{1}{2} N_i + s_1 \cdot p_i$ to obtain:

$$\begin{aligned} \Delta_3 \mathcal{P}^\mu &= \frac{1}{2} \delta_{p_1 p_2 p_3}^{s_1 \alpha \beta} \delta_{p_1 p_2 p_3}^{\mu \alpha \beta} \\ &- \frac{1}{2} (+N_1 \delta_{\mu p_2 p_3}^{p_1 p_2 p_3} - N_2 \delta_{\mu p_3 p_4}^{p_1 p_2 p_3} + N_3 \delta_{\mu p_4 p_1}^{p_1 p_2 p_3} - N_4 \delta_{\mu p_1 p_2}^{p_1 p_2 p_3}) \end{aligned} \quad (101)$$

The evaluation of \mathcal{P}^2 could also be done with the use of the above equation, but it is better to consider $Q^2 - \mathcal{P}^2$ as a single object.

The reduction of the four-point function tensors integrals usually gives no problems in the limit that the dimension of space time is taken to 4. The only exception is formed by the integral with four powers of the momenta in the numerator. This integral has a divergent part, so we have to keep track of its n dependence. In that case we have to readjust equation (91). It is no longer possible to use the vector v so we have to write

$$Q^\mu = \mathcal{P}^\mu + w_Q^\mu/\Delta_3 \quad (102)$$

with

$$w_\nu^\mu = \delta_{p_1 p_2 p_3 \nu}^{p_1 p_2 p_3 \mu} \quad (103)$$

One may note that w_ν^μ/Δ_3 is a projection operator into the space perpendicular to p_1 , p_2 and p_3 .

We consider the n -dimensional equation for $Q^\mu Q^\nu$ first. Although this is usually not necessary it illustrates the differences with the 4-dimensional case. The equivalent of the last term in equation (97) is

$$\underline{w_Q^\mu w_Q^\nu/\Delta_3^2} \quad (104)$$

This can only be proportional to $w^{\mu\nu}$. The coefficient can again be found by contracting with $g_{\mu\nu}$. (Another way of looking on this is to remember that $\underline{Q_\alpha Q_\beta}$ can also be expressed in terms proportional to $p_i^\alpha p_j^\beta$ and $g^{\alpha\beta}$. Only this last term can lead to a result that is not zero when contracted with w).

Next the contraction w_μ^μ gives the value $n - 3$. Thus the equations (97) and (98) can be modified by replacing each pair $v^\mu v_\nu$, not contracted with Q by $w^{\mu\nu}/(n - 3)$. In the only interesting case (99) we have to make one slight modification to the last term to obtain

$$\begin{aligned} \underline{Q^\mu Q^\nu Q^\rho Q^\sigma} &= \underline{\mathcal{P}^\mu \mathcal{P}^\nu \mathcal{P}^\rho \mathcal{P}^\sigma} \\ &+ \underline{(w^{\mu\nu} \mathcal{P}^\rho \mathcal{P}^\sigma + w^{\mu\rho} \mathcal{P}^\nu \mathcal{P}^\sigma + w^{\mu\sigma} \mathcal{P}^\nu \mathcal{P}^\rho + w^{\nu\rho} \mathcal{P}^\mu \mathcal{P}^\sigma + w^{\nu\sigma} \mathcal{P}^\mu \mathcal{P}^\rho + w^{\rho\sigma} \mathcal{P}^\mu \mathcal{P}^\nu)} \\ &\quad \times (Q^2 - \mathcal{P}^2)/((n - 3)\Delta_3) \\ &+ (w^{\mu\nu} w^{\rho\sigma} + w^{\mu\rho} w^{\nu\sigma} + w^{\mu\sigma} w^{\nu\rho}) \underline{(Q^2 - \mathcal{P}^2)^2}/((n - 3)(n - 1)\Delta_3^2) \end{aligned} \quad (105)$$

5.2 Three- and two-point functions

Now we are ready to attack the three-point functions. These are conceptually identical to four point functions in n dimensions. Specifically, the decomposition of the perpendicular term must be done in terms of a tensor like w_ν^μ . We will write this tensor here as a generalized Kronecker delta and keep the notation with the w reserved for the tensor belonging to the four point function. We now have

$$Q^\mu = \mathcal{P}^\mu + \delta_{p_1 p_2 Q}^{p_1 p_2 \mu}/\Delta_2 \quad (106)$$

$$\mathcal{P}^\mu = \delta_{p_1 p_2}^{\mu \alpha} \delta_{p_1 p_2}^{\alpha Q}/\Delta_2 \quad (107)$$

This leads to the following forms for the integrals (in this section $\underline{Q^\mu}$ refers to the three-point integral $\int d^4 Q Q^\mu/N_1 N_2 N_3$)

$$\underline{Q^\mu} = \underline{\mathcal{P}^\mu} \quad (108)$$

$$\underline{Q^\mu Q^\nu} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu} + \underline{\delta_{p_1 p_2 Q}^{p_1 p_2 \mu} \delta_{p_1 p_2 Q}^{p_1 p_2 \nu}}/\Delta_2^2 \quad (109)$$

This last term is proportional to $\delta_{p_1 p_2 \nu}^{p_1 p_2 \mu}$. The coefficient follows again from the contraction of μ and ν .

$$\underline{Q^\mu Q^\nu} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu} + \delta_{p_1 p_2 \nu}^{p_1 p_2 \mu} (Q^2 - \mathcal{P}^2) / ((n-2)\Delta_2) \quad (110)$$

$$\underline{Q^\mu Q^\nu Q^\rho} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu \mathcal{P}^\rho} + \frac{(\delta_{p_1 p_2 \nu}^{p_1 p_2 \mu} \mathcal{P}^\rho + \delta_{p_1 p_2 \rho}^{p_1 p_2 \nu} \mathcal{P}^\mu + \delta_{p_1 p_2 \mu}^{p_1 p_2 \rho} \mathcal{P}^\nu)(Q^2 - \mathcal{P}^2)}{((n-2)\Delta_2)} \quad (111)$$

The reduction of the \mathcal{P} to N_i and constants is analogous to the case of the four-point function and gives

$$\Delta_2 \mathcal{P}^\mu = \delta_{p_1 p_2}^{s_1 \alpha} \delta_{p_1 p_2}^{\mu \alpha} - \frac{1}{2} (N_1 \delta_{p_1 p_2}^{\mu p_2} + N_2 \delta_{p_1 p_2}^{\mu p_3} + N_3 \delta_{p_1 p_2}^{\mu p_1}) \quad (112)$$

The two-point function can be treated the same way:

$$Q^\mu = \mathcal{P}^\mu + \delta_{pQ}^{\mathcal{P}\mu} / p^2 \quad (113)$$

$$\mathcal{P}^\mu = p^\mu p \cdot Q / p^2 \quad (114)$$

so that

$$\underline{Q^\mu} = \underline{\mathcal{P}^\mu} \quad (115)$$

$$\underline{Q^\mu Q^\nu} = \underline{\mathcal{P}^\mu \mathcal{P}^\nu} + \frac{(Q^2 - \mathcal{P}^2) \delta_{p\nu}^{\mathcal{P}\mu}}{((n-1)p^2)} \quad (116)$$

and

$$p^2 \mathcal{P}^\mu = p^\mu s \cdot p - \frac{1}{2} (N_1 - N_2) p^\mu \quad (117)$$

Of course this does not make sense when $p^2 = 0$. In that case we express the tensor integrals in a sum of all possible tensor structures with undetermined coefficients:

$$B_1^\mu(0, s_1^2, s_2^2) = p^\mu B_{11} \quad (118)$$

$$B_2^{\mu\nu}(0, s_1^2, s_2^2) = p^\mu p^\nu B_{21} + g^{\mu\nu} B_{22} \quad (119)$$

These are solved for by contracting these equations with p_μ and $g_{\mu\nu}$. As many equations now vanish it is necessary to consider contractions of up to $B_3^{\mu\nu\rho}$ to obtain the coefficients of $B_2^{\mu\nu}$. We obtain (see also [7])

$$B_0(0, s_1^2, s_2^2) = \frac{A_0(s_2^2) - A_0(s_1^2)}{s_2^2 - s_1^2} \quad (120)$$

$$B_1^\mu(0, s_1^2, s_2^2) = p^\mu \left(\frac{2 s_2^2 A_0(s_2^2) - s_1^2 A_0(s_1^2)}{n (s_2^2 - s_1^2)^2} - \frac{A_0(s_2^2)}{s_2^2 - s_1^2} \right) \quad (121)$$

$$\begin{aligned} B_2^{\mu\nu}(0, s_1^2, s_2^2) = & p^\mu p^\nu \left(\frac{8 s_1^2 (s_2^2 A_0(s_2^2) - s_1^2 A_0(s_1^2))}{n(n+2) (s_2^2 - s_1^2)^3} \right. \\ & \left. + A_0(s_2^2) \left(-\frac{2}{n+2} \frac{s_2^2 + s_1^2}{(s_2^2 - s_1^2)^2} + \frac{n}{n+2} \frac{1}{s_2^2 - s_1^2} \right) \right) \\ & + g^{\mu\nu} \left(\frac{1}{n} \frac{s_2^2 A_0(s_2^2) - s_1^2 A_0(s_1^2)}{s_2^2 - s_1^2} \right) \end{aligned} \quad (122)$$

The final special case concerns not only $p^2 = 0$ but in addition we take $m_1 = m_2$. In this case we have to study tensors with four powers of Q in the numerator before we have the equations that fix the form factors in the tensors with two powers in the numerator. The result is:

$$B_0(0, s^2, s^2) = \frac{n-2}{2s^2} A_0(s^2) \quad (123)$$

$$B_1^\mu(0, s^2, s^2) = -\frac{1}{2} p^\mu B_0(0, s^2, s^2) \quad (124)$$

$$B_2^{\mu\nu}(0, s^2, s^2) = \left(\frac{1}{3} p^\mu p^\nu + \frac{s^2}{n-2} g^{\mu\nu} \right) B_0(0, s^2, s^2) \quad (125)$$

We have expressed these equations in terms of the s_i^2 rather than the masses to minimize the dependence of these equations on the choice of the metric.

5.3 Five- and higher point functions

Five and higher point tensor integrals are easier than the previous cases [12]. As they will usually occur in such a tensor combination that they are finite one may use 4-dimensional methods. It is easiest to just use the Schouten identity

$$Q^\mu \varepsilon^{p_1 p_2 p_3 p_4} = p_1 \cdot Q \varepsilon^{\mu p_2 p_3 p_4} + p_2 \cdot Q \varepsilon^{p_1 \mu p_3 p_4} + p_3 \cdot Q \varepsilon^{p_1 p_2 \mu p_4} + p_4 \cdot Q \varepsilon^{p_1 p_2 p_3 \mu} \quad (126)$$

with p_1, p_2, p_3, p_4 any independent set of momenta. This means that

$$Q^\mu = \mathcal{P}^\mu \quad (127)$$

without extra terms. The higher order tensor integrals are thus only products of this \mathcal{P} . However, one can not define internal vectors s_i for five- and higher point functions. The dotproducts $s_i \cdot p_j$ can still be seen as a short-hand notation for a combination of masses and physical dotproducts; alternatively, the s_i can be seen to exist in a higher-dimensional vector space with only the projection on the physical four-dimensional Minkowski space having significance. We thus proceed as usual to obtain

$$\begin{aligned} \mathcal{P}^\mu = & (s_1 \cdot p_1 \varepsilon^{\mu p_2 p_3 p_4} - s_1 \cdot p_2 \varepsilon^{\mu p_1 p_3 p_4} + s_1 \cdot p_3 \varepsilon^{\mu p_1 p_2 p_4} - s_1 \cdot p_4 \varepsilon^{\mu p_1 p_2 p_3}) / \varepsilon^{p_1 p_2 p_3 p_4} \\ & - \frac{1}{2} (N_1 \varepsilon^{\mu p_2 p_3 p_4} - N_2 \varepsilon^{\mu(p_1+p_2)p_3 p_4} + N_3 \varepsilon^{\mu p_1(p_2+p_3)p_4} - N_4 \varepsilon^{\mu p_1 p_2(p_3+p_4)} \\ & + N_5 \varepsilon^{\mu p_1 p_2 p_3}) / \varepsilon^{p_1 p_2 p_3 p_4} \end{aligned} \quad (128)$$

6 Conclusion

The use of the on-shell unphysical vectors s_i in combination with the use of kinematical determinants offers a compact notation and a good starting point for calculations of the scalar one-loop integrals. The arguments of the dilogarithms can be expressed directly in a small number of these determinants. Special simple cases can easily be derived from

our general formulae by just working out the determinants analytically. Cancellations among the dilogarithms have almost all been circumvented using its algebraic properties. The resulting algorithms have been implemented in a Fortran program.

The tensor integrals can also be reduced to relatively compact expressions using the determinant notation and the vectors s_i . It is very easy to program this resulting reduction scheme in a symbolic manipulation language like Form.

Nearly all our formulae are metric independent as the masses have been absorbed inside four-vectors. The main metric dependence that is left concerns the definition of the denominators and the definition of the s_i (12). There is a relative minus sign for each N_i when Pauli metric is used. In addition one must be careful about the sign of $i\epsilon$.

A Appendix: An economical reduction scheme

If the equations for the reduction of tensor integrals are used as they are given in the text they may lead to an explosive increase of the size of the expressions when the higher powers of Q are involved. It is therefore better to study the equations carefully. We here give a reduction algorithm which limits the number of terms as far as possible.

Given a tensor n -point function the following recipe is followed:

1. Replace the vectors Q by \mathcal{P} using equations (94)–(99) and their counterparts for the 3- and 2-point functions.
2. Collect the terms with $(Q^2 - \mathcal{P}^2)$ in the numerator.
3. Use equation (101), (112) or (117) to expand the \mathcal{P} as far as needed.
4. Replace the remaining \mathcal{P} by Q .

One is then left with $(n-1)$ -point tensor integrals, scalar n -point integrals and integrals with only powers of $(Q^2 - \mathcal{P}^2)$ in the numerator, and the whole scheme can be repeated for the $(n-1)$ -points tensor integrals. We will now elaborate on the different steps.

A.1 The terms with $Q^2 - \mathcal{P}^2$

After the substitution of Q by \mathcal{P} with equations (94)–(99) for the four-point functions, (108)–(111) for the three-point functions and (115)–(116) for the two-point functions we are left with terms proportional to $(Q^2 - \mathcal{P}^2)$ and to \mathcal{P} . The reduction of the first kind of terms is discussed here, the last kind terms is handled in section A.2.

The first problem we encounter is that the vectors \mathcal{P} are different for the different n -point functions. However, we note that in all equations the combination $Q^2 - \mathcal{P}^2$ is proportional to w_Q^Q , with w defined in equation (103) for the four-point function. We show this here for the three-point function:

$$\begin{aligned} \frac{w_Q^Q}{\Delta_3} &= \frac{w_\nu^\mu \delta_{p_1 p_2}^{\nu \mu} Q^2 - \mathcal{P}^2}{(n-2)\Delta_3\Delta_2} \\ &= \frac{n-3}{n-2} \frac{Q^2 - \mathcal{P}^2}{\Delta_3\Delta_2} \end{aligned} \quad (129)$$

In general one can find the relevant rewrite in terms of w_Q^Q by contracting the left hand side of the tensor equations with $w_{\mu\nu}$ (and twice with a w for the four powers of Q) and noticing that $w_\mu^\nu \mathcal{P}_\nu$ is zero in all equations.

The scheme that we like to apply now is to replace all occurrences of $Q^2 - \mathcal{P}^2$ by w_Q^Q . These factors don't need any modification when we go from higher point functions to lower point functions, whereas the vector \mathcal{P} is meant for a particular integral only.

After all substitutions have been applied these integrals can be written back in terms of $Q^2 - \mathcal{P}^2$, with the \mathcal{P} defined for that particular integral. The reduction of

those integrals is given below:

$$\int d^n Q \frac{Q^2 - \mathcal{P}^2}{N_1 N_2} = \int d^n Q \left\{ \frac{\delta^{s_1 p_1}}{p_1^2 N_1 N_2} + \frac{s_2 \cdot p_1}{2 p_1^2 N_2} - \frac{s_1 \cdot p_1}{2 p_1^2 N_1} \right\} \quad (130)$$

$$\int d^n Q \frac{Q^2 - \mathcal{P}^2}{N_1 N_2 N_3} = \int d^n Q \left\{ \frac{\delta^{s_1 p_1 p_2}}{\Delta_2 N_1 N_2 N_3} + \frac{\delta^{s_2 p_2}}{2 \Delta_2 N_2 N_3} + \frac{\delta^{s_1 (-p_1 - p_2)}}{2 \Delta_2 N_1 N_3} + \frac{\delta^{s_1 p_1}}{2 \Delta_2 N_1 N_2} \right\} \quad (131)$$

$$\int d^n Q \frac{Q^2 - \mathcal{P}^2}{N_1 N_2 N_3 N_4} = \int d^n Q \left\{ \frac{\delta^{s_1 p_1 p_2 p_3}}{\Delta_3 N_1 N_2 N_3 N_4} + \frac{\delta^{s_2 p_2 p_3}}{2 \Delta_3 N_2 N_3 N_4} - \frac{\delta^{s_1 p_3 p_4}}{2 \Delta_3 N_1 N_3 N_4} + \frac{\delta^{s_1 p_4 p_1}}{2 \Delta_3 N_1 N_2 N_4} - \frac{\delta^{s_1 p_1 p_2}}{2 \Delta_3 N_1 N_2 N_3} \right\} \quad (132)$$

and

$$\int d^n Q \frac{(Q^2 - \mathcal{P}^2)^2}{N_1 N_2 N_3 N_4} = \int d^n Q \left\{ \frac{\delta^{s_1 p_1 p_2 p_3} (Q^2 - \mathcal{P}^2)}{\Delta_3 N_1 N_2 N_3 N_4} + \frac{(n-3) \delta^{s_2 p_2 p_3} (Q^2 - \mathcal{P}^2)}{2(n-2) \Delta_3 N_2 N_3 N_4} - \frac{(n-3) \delta^{s_1 p_3 p_4} (Q^2 - \mathcal{P}^2)}{2(n-2) \Delta_3 N_1 N_3 N_4} + \frac{(n-3) \delta^{s_1 p_4 p_1} (Q^2 - \mathcal{P}^2)}{2(n-2) \Delta_3 N_1 N_2 N_4} - \frac{(n-3) \delta^{s_1 p_1 p_2} (Q^2 - \mathcal{P}^2)}{2(n-2) \Delta_3 N_1 N_2 N_3} \right\} \quad (133)$$

In this last equation each \mathcal{P} is already the proper \mathcal{P} of the integral in which it occurs. This means that one can use (131) directly for substitution in (133).

The derivation of these last four equations is done by substituting one power of \mathcal{P} after which some study of the terms with the N_i reveals that they are all zero when integrated over with the exception of one (it has to be shifted over p_1). Then the substitution of the second power of \mathcal{P} and a minor amount of rewriting gives the above equations.

A.2 The terms with \mathcal{P}

The terms with $(Q^2 - \mathcal{P}^2)$ having been reduced to scalar integrals we now turn our attention to the other terms with powers of \mathcal{P} .

In order to minimise the number of terms we reduce the tensor n-point integrals to tensor (n-1)-point integrals only and avoid lower point functions. This means that we first replace exactly one power of \mathcal{P}^μ using one of the equations (101), (112) or (117). Next the terms in which there only remain (n-1) denominators N_i left are left alone, whereas we keep substituting one power at a time in those terms that didn't obtain powers of the denominators to alter the type of the integral. Next we write the \mathcal{P} that are left back to Q via the defining equation of \mathcal{P} (the equation that decomposed Q). The reason of this manoeuvre is that we like to avoid terms of the type N_i^2 . When the integrals have been written back in terms of Q and $\mathcal{P}^2 - Q^2$ we can continue with a similar procedure at a lower level.

This reduction at the lower level gives one minor complication. If N_1 was removed when going from a n -point tensor integral to a $(n-1)$ -point tensor integral we have to shift Q before we can apply the equations in the text, as these equations can only be used if one of the denominators is of the form $(Q^2 - m^2)$.

The reduction of the two point functions is analogous with the extra complication that $p_1 \cdot p_1$ could be zero and in addition the masses of the particles in the loop could be identical. The relevant formulae were shown in the text. The one point functions are very simple.

A.3 Results

It is fairly easy to implement the above algorithms in a symbolic manipulation program. In the language Form [9] each of the procedures of the 4, 3, and 2-point takes less than one page of computer code. With these it has been proven possible to do a complete reduction of the four-point function $\underline{Q^\mu Q^\nu Q^\rho Q^\sigma}$ with all parameters different to only 1443 terms containing only scalar integrals (or 638 terms with the $(Q^2 - \mathcal{P}^2)$ terms unexpanded).

After the reduction there will be many terms involving a determinant like w or δ . It is usually best to evaluate such terms numerically, rather than to rewrite them in terms of dotproducts. The first reason is one of practicality: when they are expanded the number of terms may be multiplied by a big factor. The second reason is one of accuracy: these terms are usually in their stable form when written in terms of the uncontracted Levi-Civita tensors. One needs then only a one time investment of obtaining the accurate numbers.

When a particularly simple reaction is studied it could be beneficial to first use the above reduction scheme and then contract the Levi-Civita tensors anyway, because the contractions may become very simple. A good example of such a reaction is photon-photon scattering. This reaction has only three parameters and for instance the term $\delta_{s_1 p_1 p_2 p_3}^{s_1 p_1 p_2 p_3}$ can be reduced to three terms rather than the regular 386 terms of the most general case in which there are 10 parameters. Note also that already we needed the square root of this object in equation (82) for the evaluation of the scalar four point function.

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